J. Fluid Mech. (1984), *vol.* 149, *pp.* 59–69
Printed in Great Britain

Integral-equation solution of potential flow past a porous body of arbitrary shape

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(Received 7 November 1983 and in revised form 22 June 1984)

Potential flow past a porous body of arbitrary shape with constant physical permeability k_0, as well as the flow in the corresponding porous medium, are analysed by means of a pair of linear Fredholm integral equations of the second kind. As an example for verification of the proposed general method, the case of a two-dimensional porous circular cylinder is worked out in detail.

1. Introduction

Potential flow past a non-permeable rigid body is a very well-known subject; however, ignorance has reigned when the body is permeable. In this paper the problem of determining the potential flow past a porous body of arbitrary shape with physical permeability constant k_0, as well as the flow in the corresponding porous medium, are analysed. The flow exterior to the body is subject to an asymptotically constant prescribed velocity at infinity, and the corresponding potential function is postulated to be a linear combination of two potentials. One represents the flow past a solid body with the same geometrical configuration as the given porous body, whose asymptotic behaviour at infinity coincides with that of the overall flow. The other potential is defined to behave at infinity in such a way that its gradient tends to zero faster than the corresponding velocity of a source located inside the body, since the net flux across the porous body has to be zero; in other words, all the fluid mass coming into the porous body will return to the exterior flow. The flow interior to the body is represented as a potential flow with the corresponding pressure related to the seepage velocity by Darcy's law.

The first exterior potential flow corresponding to flow past a solid body is determined by means of a linear Fredholm integral equation of the second kind in a standard manner. The flux-matching condition on the porous-body surface allows the representation of both the interior potential flow and the second exterior potential flow as one double-layer potential defined throughout all space, whose density is determined from a nonlinear integral equation resulting from the pressure-matching condition. A formal solution to this nonlinear equation is found in terms of the solution to a certain linear integral equation when a dimensionless parameter K^* is small. A discussion of d'Alembert's paradox and the force on a porous obstacle is also presented.

As a check on the proposed general method, the case of two-dimensional flow around a porous circular cylinder is worked out in detail.

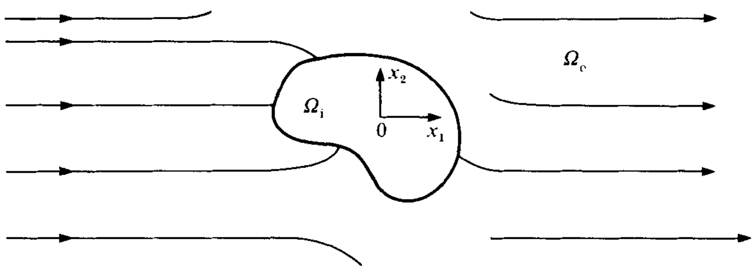

FIGURE 1

2. Integral-equation solution of the problem

Consider a porous body B with physical permeability constant K_0 occupying a three-dimensional domain Ω_i bounded by a closed Liapunov surface S such as defined in Gunter (1967, p. 1), and let Ω_e be the complement of $\bar{\Omega}_i$, as shown in figure 1.

Assume that throughout Ω_e there exists a potential flow ('exterior flow') of a constant density ρ, with a prescribed uniform asymptotic behaviour at infinity. Hence the potential ϕ describing the flow Ω_e must satisfy

$$\nabla^2\phi(x_1, x_2, x_3) = 0 \quad \text{for every point } x \in \Omega_e, \tag{1}$$

$$\lim_{r\to\infty} \nabla\phi(x_1, x_2, x_3) = (U, 0, 0) \quad \text{uniformly}, \tag{2}$$

where $r = (x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}$, U is a given scalar velocity, and (x_1, x_2, x_3) are Cartesian coordinates with fixed origin O chosen inside the body. Since the problem under consideration deals with a fluid of constant density in an enclosed system, without free surface, the dynamic pressure P_e in Ω_e is given by Bernoulli's law as

$$P_e(x) = \tfrac{1}{2}\rho\{U^2 - [\nabla\phi(x)]^2\} \quad \text{for every point } x \in \Omega_e. \tag{3}$$

Part of the flow in Ω_e seeps through the porous body B, entering and exiting through the surface pores on S. The flow seeping through B will be represented as a potential flow, with the corresponding dynamic pressure P_i related to the seepage velocity $\nabla\phi_i$ by Darcy's Law, and thus the potential ϕ_i describing the flow in Ω_i has to satisfy

$$\nabla^2\phi_i(x_1, x_2, x_3) = 0 \quad \text{for every point } x \in \Omega_i \tag{4}$$

and

$$\nabla P_i(x_1, x_2, x_3) = -\frac{\mu}{k_0}\nabla\phi_i(x_1, x_2, x_3), \tag{5}$$

or equivalently

$$P_i(x_1, x_2, x_3) = -\frac{\mu}{k_0}\phi_i(x_1, x_2, x_3) \quad \text{for every point } x \in \Omega_i. \tag{6}$$

The flow state $(\nabla\phi_i, P_i)$ will be called the 'interior porous flow' from now on.

On the surface S (of the porous body) it can be said that the dynamic pressure does not experience a jump as the flow enters or exits the body B through S; and, assuming that there are no sources or sinks of fluid on S, the following *matching conditions* for both flux and pressure must hold:

$$\left.\begin{aligned} \boldsymbol{n}(x^*)\cdot\nabla\phi(x^*) &= \boldsymbol{n}(x^*)\cdot\nabla\phi_i(x^*), && (7)\\ \lim_{x\to x^*}\tfrac{1}{2}\rho\{[\nabla\phi(x)]^2 - U^2\} &= \lim_{x\to x^*}\frac{\mu}{k_0}\phi_i(x) && (8)\end{aligned}\right\} \text{ for every point } x^* \in S,$$

where $\boldsymbol{n}(x^*)$ is the normal to S at x^* directed into Ω_e, and the dot indicates the usual scalar product in three-dimensional space. The products appearing in (7) must of course be understood as limiting values of the directional derivatives of ϕ and ϕ_i in the direction $\boldsymbol{n}(x^*)$ as the point of evaluation tends to x^*.

In order to find functions ϕ and ϕ_i satisfying (1), (2), (4), (7) and (8), we shall express ϕ as a linear combination of two auxiliary potential functions ϕ_1 and ϕ_e'':

$$\phi = \phi_1 + K^*\phi_e'', \tag{9}$$

where $K^* = k_0\rho U/\mu D$ is a new dimensionless physical parameter, which will be assumed to be much less than unity in order to linearize the matching condition (8), and D is a representative dimension of the body.

ϕ_1 is taken to be the usual potential flow around a solid body of the same geometrical configuration as that of the given porous body B, and thus ϕ_1 satisfies

$$\nabla^2\phi_1(x) = 0 \quad \text{for all } x \in \Omega_e, \tag{10}$$

$$\lim_{x\to\infty} \nabla\phi_1(x) = (U, 0, 0) \quad \text{uniformly}, \tag{11}$$

$$\boldsymbol{n}(x)\cdot\nabla\phi_1(x) = 0 \quad \text{for all } x \in S. \tag{12}$$

The problem defined by (10)–(12) is readily found to reduce to an ordinary exterior Neumann problem with vanishing gradient at infinity for an auxiliary potential function ϕ_1' by simply writing

$$\phi_1(x) = Ux_1 + \phi_1'(x). \tag{13}$$

Obviously, the potential function ϕ_1' must satisfy the following Neumann boundary condition:

$$\boldsymbol{n}(x^*)\cdot\nabla\phi_1'(x^*) = -Un_1(x^*) \quad \text{for all } x^* \in S, \tag{14}$$

where $\boldsymbol{n} = (n_1, n_2, n_3)$.

The above boundary condition, together with the vanishing asymptotic behaviour of $\nabla\phi_1'$ at infinity, are sufficient conditions to guarantee the uniqueness of potential functions ϕ_1' solving this exterior Neumann problem up to additive constants.

Since single-layer surface potentials with continuous density spread over a Liapunov surface S (as defined by (15) below) do have a vanishing gradient at infinity, are continuous throughout space, and their normal derivatives experience a jump across S, it is most natural, in the light of Fredholm's integral-equation approach for the solution of boundary-value problems, to seek the solution ϕ_1' as one such single layer with unknown density $(-\tilde{\phi})/2\pi$:

$$\phi_1'(x) = -\frac{1}{2\pi}\int_S \frac{\tilde{\phi}(y)}{r_{xy}}\,\mathrm{d}S_y, \tag{15}$$

where $\mathrm{d}S_y$ is the surface-area element at point $y = (y_1, y_2, y_3) \in S$, and

$$r_{xy}^2 = \sum_{i=1}^{3} (x_i - y_i)^2.$$

Using the well-known limiting value as the point x tends to a point $x^* \in S$ of the directional derivative of a single-layer potential in the direction $\boldsymbol{n}(x^*)$ (see Gunter 1967, p. 125), the following Fredholm integral equation of the second kind is found for the unknown continuous density $\tilde{\phi}$ by virtue of (14):

$$\tilde{\phi}(x^*) - \frac{1}{2\pi}\int_S \frac{\tilde{\phi}(y)\cos(\boldsymbol{r}_{x^*y}, \boldsymbol{n}(x^*))}{r^2_{x^*y}}\,\mathrm{d}S_y = -Un_1(x^*) \tag{16}$$

for all $x^* \in S$, where $\boldsymbol{r}_{x^*y}$ is the vector directed from x^* to y.

The above non-homogeneous linear equation of the second kind, being the adjoint of the integral equation corresponding to the interior Dirichlet problem, was shown to have a unique solution in the two-dimensional case by Fredholm (1900) and by Gunter (1967, p. 152) in the present situation of Liapunov surfaces S embedded in Euclidean three-dimensional space.

The analytical solution of (16) is given by means of the following uniformly convergent modified Neumann series for $\tilde{\phi}$ (see Gunter 1967, pp. 152, 127):

$$\tilde{\phi}(x^*) = -\tfrac{1}{2}[\tilde{\phi}_0(x^*) - (\tilde{\phi}_1(x^*) - \tilde{\phi}_0(x^*)) + (\tilde{\phi}_2(x^*) - \tilde{\phi}_1(x^*))\ldots], \tag{17}$$

where the terms of the series (17) are calculated recursively at $x^* \in S$ by the following formulae:

$$\left.\begin{aligned} \tilde{\phi}_0(x^*) &= -Un_1(x^*), \\ \tilde{\phi}_m(x^*) &= -\frac{1}{2\pi}\int_S \tilde{\phi}_{m-1}(y)\frac{\cos(\boldsymbol{r}_{x^*y}, \boldsymbol{n}(x^*))}{r^2_{x^*y}}\,\mathrm{d}S_y \end{aligned}\right\} \text{ for } m \neq 0. \tag{18}$$

The above series can be used as a basis for a numerical solution with a reasonable efficiency, profiting from the recursive character of the integrals to be evaluated, much in the same manner as the successive approximations solution discussed by many authors like Swarztrauber (1973) and Chow, How & Landweber (1976). Alternatively, (16) could be solved numerically in a direct manner by means of discretization of the surface integral, leading to an algebraic linear system of equations.

The second auxiliary potential function ϕ''_e is defined to behave at infinity in such a way that its gradient tends to zero faster than r^{-2}, where r is the distance from the origin inside the body and a point $x \in \Omega_e$, since the source behaviour at infinity is not possible because the matching condition (7) implies no net flux across the surface of the porous body. Hence

$$\nabla^2\phi''_e(x) = 0 \quad \text{for all } x \in \Omega_e, \tag{19}$$

$$\boldsymbol{\nabla}\phi''_e(x) \sim \frac{1}{r^n} \quad \text{with } n > 2 \text{ for large } r. \tag{20}$$

It can be seen that ϕ defined by (9) satisfies (1) because of (10) and (19), and satisfies (2) because of (11) and (20).

Obviously, (19) and (20) do not suffice to determine ϕ''_e, since the boundary values of ϕ''_e on S are not prescribed, and likewise the boundary values of ϕ_i are not known beforehand. It will be seen presently that the matching conditions (7) and (8) actually determine such boundary values of ϕ''_e and ϕ_i.

Because of (9) and (12) the following must hold:

$$\boldsymbol{n}(x^*)\cdot\boldsymbol{\nabla}\phi(x^*) = \boldsymbol{n}(x^*)\cdot\boldsymbol{\nabla}K^*\phi''_e(x^*) \quad \text{for all } x^* \in S. \tag{21}$$

Considering (7) and (21), it is convenient to introduce in Ω_i another auxiliary potential function ϕ''_i such that

$$\phi_i = K^*\phi''_i. \tag{22}$$

From (4), (22), (7) and (21), one can see that ϕ_i'' must satisfy

$$\nabla^2\phi_i''(x) = 0 \quad \text{for all } x \in \Omega_i, \tag{23}$$

$$\boldsymbol{n}(x^*)\cdot\boldsymbol{\nabla}\phi_i''(x^*) = \boldsymbol{n}(x^*)\cdot\boldsymbol{\nabla}\phi_e''(x^*) \quad \text{for all } x^* \in S. \tag{24}$$

Because of (24), ϕ_i'' and ϕ_e'' can be thought of as continuing each other across S in such a way that their normal derivatives do not experience a jump across S. In the light of Liapunov's theorem on the continuity of normal derivatives of double-layer potentials (see Gunter 1967, p. 297), the no-jump condition (24) suggests the choice of ϕ_i'' and ϕ_e'' as restrictions to Ω_i and Ω_e respectively of a double-layer potential W with unknown density $\nu/2\pi$ spread over S, i.e.

$$\phi_{i,e}''(x) = W(x) = \frac{1}{2\pi}\int_S \frac{\nu(y)\cos(\boldsymbol{r}_{xy}, \boldsymbol{n}(y))\,\mathrm{d}S_y}{r_{xy}^2}, \tag{25}$$

where the subscripts i and e are used when $x \in \Omega_i$ or $x \in \Omega_e$ respectively.

The choice automatically satisfies (19), (20), (4) and the matching condition (7) when ν is sufficiently regular by virtue of the abovementioned Liapunov's theorem, so that, in order to solve completely the problem, it only remains to satisfy the pressure-matching condition (8).

By virtue of (9) and (22), the matching condition (8) can be written as

$$\lim_{x\to x^*} (\boldsymbol{\nabla}(\phi_1 + K^*\phi_e''))^2(x) - U^2 = \frac{2U}{D}\lim_{x\to x^*}\phi_i''(x) \quad \text{for all } x^* \in S. \tag{26}$$

Using the well-known limiting values at S of double-layer potentials with continuous densities spread over Liapunov surfaces (see Gunter 1967, p. 49) given by (27) and (28) below, it can be seen after their substitution into (26) that this pressure-matching condition amounts to a nonlinear integral equation for the unknown density ν, which we shall linearize for small values of the parameter K^*:

$$\lim_{x\to x^*}\phi_i''(x) = \frac{1}{2\pi}\int_S \frac{\nu(y)\cos(\boldsymbol{r}_{x^*y}, \boldsymbol{n}(y))}{r_{x^*y}^2}\,\mathrm{d}S_y + \nu(x^*), \tag{27}$$

$$\lim_{x\to x^*}\phi_e''(x) = \frac{1}{2\pi}\int_S \frac{\nu(y)\cos(\boldsymbol{r}_{x^*y}, \boldsymbol{n}(y))}{r_{x^*y}^2}\,\mathrm{d}S_y - \nu(x^*). \tag{28}$$

Since both ϕ_i'' and ϕ_e'' are functions of the same unknown density ν, it follows that, if (26) is satisfied, then ν and all the more ϕ_i'' and ϕ_e'', must be functions of the parameter K^*. Then perturbation solutions can be assumed for ν, ϕ_i'', ϕ_e'' in terms of the small parameter K^*:

$$\nu = \nu_0 + K^*\nu_1 + (K^*)^2\nu_2 + \dots, \tag{29}$$

$$\phi_i'' = \phi_{i0}'' + K^*\phi_{i1}'' + (K^*)^2\phi_{i2}'' + \dots, \tag{30}$$

$$\phi_e'' = \phi_{e0}'' + K^*\phi_{e1}'' + (K^*)^2\phi_{e2}'' + \dots. \tag{31}$$

It should be noted that, under fairly general conditions upon the density $\tilde{\phi}$ (like Holder continuity), the single-layer potential $\phi_1' = \phi_1 - Ux_1$ possesses limiting tangential derivatives at points of S (Gunter 1967, p. 68), and this fact together with (16) assures the existence of the limiting value of $\boldsymbol{\nabla}\phi_1$ at points of S, because the right-hand side of the integral equation (16) for the density $\tilde{\phi}$ is Holder continuous on S since S is a Liapunov surface, the same is true of its solution $\tilde{\phi}$ (Gunter 1967, p. 61).

Substituting (27) into (26) results in the following linear Fredholm integral equation of the second kind for the zeroth-order approximation ν_0 to the unknown density ν:

$$\nu_0(x^*)+\frac{1}{2\pi}\int_S \frac{\nu_0(y)\cos(\boldsymbol{r}_{x^*y},\boldsymbol{n}(y))}{r^2_{x^*y}}\,\mathrm{d}S_y = f(x^*) \tag{32}$$

for all $x^*\in S$, where the function f is defined on S by

$$f(x^*) = -\tfrac{1}{2}UD+\frac{D}{2U}\lim_{x\to x^*}(\nabla\phi_1)^2(x). \tag{33}$$

The right-hand side of the non-homogeneous Fredholm integral equation (32), defined by (33), is completely known after explicitly determining ϕ_1, and (32) has a unique solution ν_0 for arbitrary continuous right-hand side.

The analytical solution of (32) is given by means of the following uniformly convergent modified Neumann series for ν_0 (see Gunter 1967, pp. 199, 180):

$$\nu_0(x^*) = \tfrac{1}{2}[\nu_0^1(x^*)-(\nu_0^2(x^*)-\nu_0^1(x^*))+(\nu_0^3(x^*)-\nu_0^2(x^*))-\dots], \tag{34}$$

where the terms of this series are calculated recursively at $x^*\in S$ by the formulae

$$\nu_0^n(x^*) = \frac{1}{2\pi}\int_S \frac{\nu_0^{n-1}(y)\cos(\boldsymbol{r}_{x^*y},\boldsymbol{n}(y))}{r^2_{x^*y}}\,\mathrm{d}S_y, \tag{35}$$

where $\nu_0^1(x^*) = f(x^*)$, and n is a positive integer.

The remarks concerning the numerical solution of (16) apply also to (32).

With the solution ν_0, the zeroth-order approximation potentials ϕ''_{i0} and ϕ''_{e0} are given by

$$\left.\begin{matrix}\phi''_{i0}(x)\\ \phi''_{e0}(x)\end{matrix}\right\} = \frac{1}{2\pi}\int_S \nu_0(y)\frac{\cos(\boldsymbol{r}_{xy},\boldsymbol{n}(y))}{r^2_{xy}}\,\mathrm{d}S_y. \tag{36}$$

Higher-order approximation potentials can be computed by a similar scheme.

Summarizing, having found ϕ''_i (and also ϕ''_e having the same dipole density) solving linear integral equations like (32), and ϕ_1 by solving the linear integral equation (16), the potential function ϕ for the exterior flow is obtained through (9), and ϕ_i, the potential function for the interior porous flow, through (22).

The preceding formulation can be extended immediately to non-constant prescribed gradient fields at infinity with zero divergence, for which the corresponding dynamic pressure at infinity is given by Bernoulli's law. Also the extension to the case of flows past a finite number of porous bodies, all of them with the same permeability constant, does not present new theoretical difficulties in this integral-equation approach.

Finally, the analogous two-dimensional problem is treated formally in the same way, considering distributions of sources and dipoles over the boundary curves of the two-dimensional regions corresponding to the porous media, recalling that the two-dimensional source is given by $\log r$ instead of $1/r$, and the two-dimensional dipole is given by $\cos(\boldsymbol{r}_{xy},\boldsymbol{n}(y))/r_{xy}$ instead of $\cos(\boldsymbol{r}_{xy},\boldsymbol{n}(y))/r^2_{xy}$.

3. On d'Alembert's paradox

The total force $\boldsymbol{F}$ exerted by a uniform inviscid potential flow surrounding a porous body arises from the exterior pressure P_e at the body surface, so that the total force will be the following surface integral:

$$\boldsymbol{F} = \tfrac{1}{2}\rho\int_S(\nabla\phi)^2\boldsymbol{n}\,\mathrm{d}S, \tag{37}$$

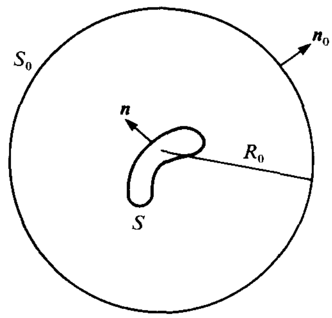

FIGURE 2

where (3) has been used to relate the exterior pressure with the exterior velocity field and also the fact that the surface integral of $U^2\boldsymbol{n}$ is zero for any closed surface.

Substituting (9) into (37), the total force can be written as

$$\boldsymbol{F} = \rho K^* \int_S \nabla\phi_e'' \cdot \nabla\phi_1 \boldsymbol{n}\, \mathrm{d}S + O(K^{*2}), \tag{38}$$

since the surface integral of $(\nabla\phi_1)^2\boldsymbol{n}$ is zero, because this integral corresponds to the total force exerted by a uniform potential flow on a solid body of the same geometrical configuration as that of the given porous body, and this force is known to be zero (d'Alembert's paradox).

Figure 2 shows a porous body of arbitrary shape; the surface of the body is denoted by S and the unit outward normal to S, locally, is denoted by $\boldsymbol{n}$. A spherical control surface S_0 of radius R_0 is set up around the body, and $\boldsymbol{n}_0$ is the unit outward normal to S_0.

Applying Gauss' theorem on the volume V, bounded by S_0 and S, the ith component of the total force is given by:

$$F_i = \rho K^* \int_{S_0} \left(\frac{\partial\phi_e''}{\partial x_j}\frac{\partial\phi_1}{\partial x_j}\right) n_{0i}\, \mathrm{d}S - \rho K^* \int_V \frac{\partial}{\partial x_i}\left(\frac{\partial\phi_e''}{\partial x_j}\frac{\partial\phi_1}{\partial x_j}\right) \mathrm{d}V + O(K^{*2}). \tag{39}$$

The volume integral in the above equation can be decomposed as the sum of two integrals:

$$\int_V \frac{\partial}{\partial x_i}\left(\frac{\partial\phi_e''}{\partial x_j}\frac{\partial\phi_1}{\partial x_j}\right)\mathrm{d}V = \int_V \frac{\partial}{\partial x_j}\left(\frac{\partial\phi_e''}{\partial x_i}\frac{\partial\phi_1}{\partial x_j}\right)\mathrm{d}V + \int_V \frac{\partial}{\partial x_j}\left(\frac{\partial\phi_e''}{\partial x_j}\frac{\partial\phi_1}{\partial x_i}\right)\mathrm{d}V, \tag{40}$$

where the irrotational character of both flows has been used. Again applying Gauss' theorem to the right-hand side of (40) and substituting into (39), the expression for the force F_i becomes:

$$F_i = \rho K^* \int_{S_0} \left(\frac{\partial\phi_e''}{\partial x_j}\frac{\partial\phi_1}{\partial x_j} n_{0i} - \frac{\partial\phi_e''}{\partial x_i}\frac{\partial\phi_1}{\partial x_j} n_{0j} - \frac{\partial\phi_e''}{\partial x_j}\frac{\partial\phi_1}{\partial x_i} n_{0j}\right) \mathrm{d}s + \int_S \frac{\partial\phi_1}{\partial x_i}\frac{\partial\phi_e''}{\partial x_j} n_j\, \mathrm{d}S + O(K^{*2}), \tag{41}$$

where the zero Neumann boundary condition for ϕ_1 $((\partial\phi_1/\partial x_j)\, n_j = 0)$ has been used.

Using the behaviour of the gradients at infinity for both potentials, where ϕ_1 was defined to behave at infinity in such a way that its gradient tends to a constant value $(U,\ 0, 0)$ and the gradient of ϕ_e'' tends to zero like r^{-3} (since ϕ_e'' is a double-layer

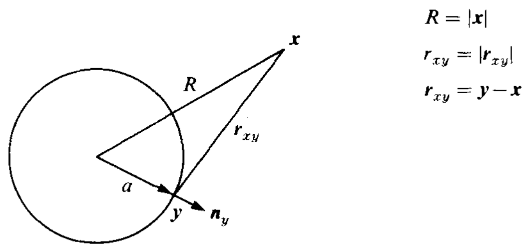

FIGURE 3

potential), it can be concluded that when R_0 tends to infinity the first surface integral in (41) tends to zero. Then

$$F_i = \rho K^* \int_S \frac{\partial \phi_1}{\partial x_i}\frac{\partial \phi_e}{\partial x_j} n_j \, \mathrm{d}S + O(K^{*2}). \tag{42}$$

Since $(\partial \phi_e''/\partial x_j)\, n_j$ on the body surface S is generally different from zero, and it is part of the solution of the present problem, it is not possible to determine beforehand from (42) the existence or non-existence of the hydrodynamic force exerted on the body by a uniform potential flow without a complete knowledge of the exterior velocity field for a given problem. However, one expects the existence of the force, because the internal flow is the seepage of a viscous fluid.

4. Two-dimensional potential flow past a porous circular cylinder

In order to verify the above general method of solution for the potential flow about an arbitrary porous body in otherwise-uniform flow, based on a pair of Fredholm integral equations, the case of a two-dimensional flow past a porous circular cylinder will be solved, since it is one of the few cases where the proposed integral equations admit simple closed-form solutions. In this case ϕ_1' will correspond to the potential of a two-dimensional irrotational flow due to a moving solid circular cylinder:

$$\phi_1'(x) = U\frac{a^2}{R}\cos\theta. \tag{43}$$

And ϕ_1 is given by

$$\phi_1(x) = UR\cos\theta\left(1+\frac{a^2}{R^2}\right), \tag{44}$$

where R is the radial distance from the centre of the cylinder to an exterior point x, and θ the corresponding polar angle. The difference between R and the distance r_{xy} that appears in the surface potentials is shown in figure 3.

The gradient of ϕ_1 on the surface of the cylinder is

$$(\nabla\phi_1)_{R=a} = (2U\sin^2\theta,\ -2U\sin\theta\cos\theta). \tag{45}$$

The proposed potentials to the first order of approximation in $K^* = k_0\rho U/\mu a$ for ϕ_i'' and ϕ_e'' at any point $x \in \Omega_i$ and $x \in \Omega_e$ respectively, given by (25), can be written here as

$$\left.\begin{matrix}\phi_{i0}''(R,\theta)\\ \phi_{e0}''(R,\theta)\end{matrix}\right\} = \frac{1}{\pi}\int_0^{2\pi} \nu_0(\varphi)\,\frac{\cos(\boldsymbol{r}_{xy}, \boldsymbol{n}(y))}{r_{xy}}\, a\,\mathrm{d}\varphi, \tag{46}$$

where x is a fixed point $(R\cos\theta,\ R\sin\theta)$ in the exterior or interior domains according to whether ϕ_{e0}'' or ϕ_{i0}'' are taken into account respectively, and y is a moving point

$(a\cos\varphi, a\sin\varphi)$ on the surface of the cylinder with outward normal $(\cos\varphi, \sin\varphi)$ at y. Therefore the kernel in equation (46) can be written as:

$$\frac{\cos(\boldsymbol{r}_{xy}, \boldsymbol{n}(y))}{r_{xy}} = \frac{1}{a}\left(\frac{1-R'\cos(\theta-\varphi)}{1+R'^2-2R'\cos(\theta-\varphi)}\right), \tag{47}$$

where $R' = R/a$; at the point $x = x^*$, with x^* on the body surface, the above relation for the kernel becomes

$$\left.\frac{\cos(\boldsymbol{r}_{x^*y}, \boldsymbol{n}(y))}{r_{x^*y}}\right|_{R'=1} = \frac{1}{2a}. \tag{48}$$

Substituting (45) and (48) into the two-dimensional version of (32), the following integral equation for $\nu_0(\theta)$ is obtained:

$$\nu_0(\theta) + \frac{1}{2\pi}\int_0^{2\pi} \nu_0(\varphi)\,\mathrm{d}\varphi = f(\theta), \tag{49}$$

where

$$f(\theta) = -\tfrac{1}{2}Ua + 2Ua\sin^2\theta. \tag{50}$$

Integrating (49) between the limits $(0, 2\pi)$, it is found that

$$\int_0^{2\pi} \nu_0(\varphi)\,\mathrm{d}\varphi = \frac{1}{2}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta, \tag{51}$$

which leads to an explicit solution of (49):

$$\nu_0(\theta) = f(\theta) - \frac{1}{4\pi}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta. \tag{52}$$

Substituting (50) into (52), there results

$$\nu_0(\theta) = -\tfrac{3}{4}Ua + 2Ua\sin^2\theta. \tag{53}$$

Substituting (53) together with (47) into (46) for a point $x = (R, \theta) \in \Omega\mathrm{e}$, one gets

$$\phi''_{e0}(R, \theta) = I_1(R, \theta) + I_2(R, \theta), \tag{54}$$

where

$$I_1(R, \theta) = \frac{2Ua}{\pi}\int_0^{2\pi} \frac{1-R'\cos(\theta-\varphi)}{1+R'^2-2R'\cos(\theta-\varphi)}\sin^2\varphi\,\mathrm{d}\varphi,$$

$$I_2(R, \theta) = -\frac{3}{4}\frac{Ua}{\pi}\int_0^{2\pi} \frac{1-R'\cos(\theta-\varphi)}{1+R'^2-2R'\cos(\theta-\varphi)}\mathrm{d}\varphi \equiv 0,$$

since the last integral is equal is equal to the plane angle subtended by the circumference $R = a$ at a point x, and when the point x is outside a closed curve, as in the present case, this angle is identically equal to zero. To evaluate I_1 the following change of variables will be used:

$$z = \mathrm{e}^{-\mathrm{i}(\theta-\varphi)}, \quad \mathrm{d}\varphi = \frac{\mathrm{d}z}{\mathrm{i}z}.$$

Therefore

$$I_1 = \frac{Ua}{4\pi\mathrm{i}}\oint \frac{\{2z^2 - \cos 2\theta(z^4+1) - \mathrm{i}\sin 2\theta(z^4-1)\}(2z - R'z^2 - R')\,\mathrm{d}z}{z^3(z + R'^2 z - R'z^2 - R')}, \tag{55}$$

where the integration is carried out around the unit circle in the complex plane. By

Cauchy's residue theorem I_1 is equal to

$$I_1 = \frac{Ua}{2} \sum_{k=1}^{k=n} \operatorname{Res}(a_k), \tag{56}$$

where $\operatorname{Res}(a_k)$ are the residues of the rational complex-function integrand of I_1 at the poles a_k; this integrand has a pole of order 3 at $z = 0$ and simple pole at $z = 1/R'$ inside the unit circle, since

$$z + R'^2 z - R'z^2 - R' = R'(R'-z)\left(z - \frac{1}{R'}\right) \quad \text{and} \quad \frac{1}{R'} < 1$$

when $x \in \Omega_e$ $(R > a)$.

After evaluating the residues, (56) becomes

$$I_1(R,\theta) = Ua^3 \frac{\cos 2\theta}{R^2} \quad (R > a). \tag{57}$$

Therefore

$$\phi''_{e0}(R,\theta) = Ua \frac{a^2}{R^2} \cos 2\theta. \tag{58}$$

Also, substituting (53) together with (47) into (46) for a point $x \in \Omega_i$, one obtains

$$\phi''_{i0}(R,\theta) = J_1(R,\theta) + J_2(R,\theta), \tag{59}$$

where

$$J_1(R,\theta) = I_1(R,\theta)$$

and

$$J_2(R,\theta) = -\frac{3}{4} \frac{Ua}{\pi} \int_0^{2\pi} \frac{1 - R'\cos(\theta-\varphi)}{1 + R'^2 - 2R'\cos(\theta-\varphi)} \, d\varphi = -\tfrac{3}{2} Ua,$$

since the plane angle equals 2π when the point x is inside the closed boundary curve, i.e. for $x \in \Omega_i$. Hence $J_1(R,\theta)$ is given by (55), but the function to be integrated this time has a simple pole at $z = R'$, since $R' < 1$ when $x \in \Omega_i$ $(R < a)$, and the same pole at $z = 0$; with this change in the simple pole, the evaluation of the residues change in such manner that $J_1(R,\theta)$ gives

$$J_1(R,\theta) = -Ua \frac{R^2}{a^2} \cos 2\theta + 2Ua \quad (R < a). \tag{60}$$

Therefore

$$\phi''_{i0}(R,\theta) = \tfrac{1}{2} Ua - Ua \frac{R^2}{a^2} \cos 2\theta. \tag{61}$$

Substituting (44), (58) and (61) into (9) and (22), one finally gets

$$\phi(x) = UR\cos\theta\left(1 + \frac{a^2}{R^2}\right) + K^* Ua \frac{a^2}{R^2} \cos 2\theta + O(K^{*2}) \tag{62}$$

for $x \in \Omega_e$, and

$$\phi_i(x) = K^* Ua\left(\frac{1}{2} - \frac{R^2 \cos 2\theta}{a^2}\right) + O(K^{*2}) \tag{63}$$

for $x \in \Omega_i$, solutions that are harmonic outside and inside the circular cylinder respectively; ϕ tends to uniform flow at infinity, and it is easy to prove that (62) and (63) satisfy the matching conditions (7) and (8) as a first approximation:

$$\left.\frac{\partial \phi}{\partial R}\right|_{R=a} = -2K^* U \cos 2\theta + O(K^{*2}),$$

$$\left.\frac{\partial \phi_i}{\partial R}\right|_{R=a} = -2K^* U \cos 2\theta + O(K^{*2})$$

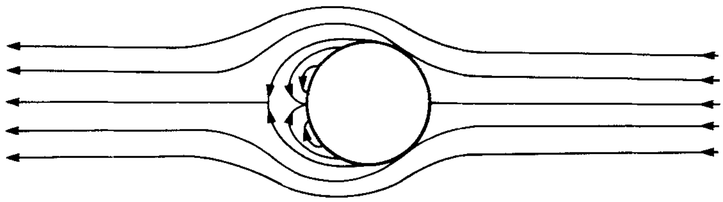

FIGURE 4

and
$$\frac{\mu}{K_0}\phi_i\bigg|_{R=a} = \rho U^2(\tfrac{1}{2}-\cos 2\theta)+O(K^*),$$
$$\tfrac{1}{2}\rho((\nabla\phi)^2-U^2)|_{R=a} = \tfrac{1}{2}\rho(4U^2\sin^2\theta-U^2)+O(K^*)$$
$$= \rho U^2(\tfrac{1}{2}-\cos 2\theta)+O(K^*).$$

As one can see, the exterior potential given by (62) consists of the sum of a uniform flow, a dipole (doublet potential) and a quadrupole. The flow pattern for the exterior flow is similar to the one shown in figure 4.

The above solutions for ϕ and ϕ_i given by (62) and (63) respectively can be found in an elementary way using cylindrical harmonics, profiting from the nice boundary geometry of the present problem. The awkwardness of the approach used in this section compared with the cylindrical-harmonics solution is natural owing to the generality of the proposed method, which sets no major restrictions on the body shape.

In order to evaluate the hydrodynamic force exerted on the porous body by the exterior potential flow, one substitutes (45) and (58) into (42) and, letting $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ denote unit vectors along the positive x- and y-axes, one finds

$$\boldsymbol{F} = K^*\rho\left\{-4U^2a\int_0^{2\pi}\sin^2\theta\cos 2\theta\,\mathrm{d}\theta\,\boldsymbol{e}_x+4U^2a\int_0^{2\pi}\sin\theta\cos\theta\cos 2\theta\,\mathrm{d}\theta\,\boldsymbol{e}_y\right\} \quad (64)$$

or
$$\boldsymbol{F} = \frac{2\pi\rho}{\nu}K_0U^3\boldsymbol{e}_x,$$

with $\nu = \mu/\rho$.

Therefore only a drag force is exerted on the porous cylinder, as is to be expected from physical considerations, and this force is independent of the size of the cylinder. This remarkable fact, independence of the force on the cylinder size, is common to the well-known case of uniform potential flow past a two-dimensional non-permeable cylinder plus an arbitrary vortex located inside the cylindrical body with intensity $\Gamma/2\pi$, where the combined flow exerts a lift force F_L given by Kutta–Joukowski as $F_L = \rho U\Gamma$.

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